Functional Analysis

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Lecture 9

Orthonormal basis

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H – fixed Hilbert space.

Def: A system of vectors $\{e_i\}_{i\in I} \subseteq H$ is called

- **orthogonal**, if $e_i \perp e_j$ for $i \neq j$ (i.e. $\langle e_i, e_j\rangle = 0$, $i \neq j$).
- **orthonormal**, if $e_i \perp e_j$, for $i \neq j$, and $\|e_i\| = 1$, for all i, that is

$$\langle e_i, e_j\rangle = \delta_{i,j} := \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \qquad i, j \in I.$$

Every orthogonal system of nonzero vectors $\{u_i\}_{i\in I}$ can be normalized to an orthonormal system $\{e_i\}_{i\in I}$ by putting $e_i := \frac{u_i}{\|u_i\|}$:

$$\langle e_i, e_j\rangle = \left\langle \frac{u_i}{\|u_i\|}, \frac{u_j}{\|u_j\|} \right\rangle = \frac{\langle u_i, u_j\rangle}{\|u_i\|\|u_j\|} = \delta_{i,j} \cdot \frac{\langle u_i, u_i\rangle}{\|u_i\|\|u_i\|} = \delta_{i,j}.$$

Prop. (Gram-Schmidt orthogonalization)

If $\{x_i\}_{i=1}^n \subseteq H$ are linearly independentnt and $M = \operatorname{lin}\{x_1, \ldots, x_n\}$, then

$$u_1 := x_1, \quad u_2 := x_2 - P_{u_1} x_2, \quad \ldots, \quad u_n := x_n - \sum_{i=1}^{n-1} P_{u_i} x_n$$

form an orthogonal system $\{u_i\}_{i=1}^n$ such that $M = \operatorname{lin}\{u_1, \ldots, u_n\}$.

Prop. If $M = \text{lin}\{e_1, ..., e_n\}$, where $\{e_i\}_{i=1}^n$ is an orthogonal system, then

$$P_M x = \sum_{i=1}^{n} \langle x, e_i \rangle e_i, \qquad \text{for every } x \in H.$$

Moreover, $\|P_M x\|^2 = \sum_{i=1}^n |\langle x, e_i\rangle|^2$ dla $x \in H$.

Proof: Let $x \in H$. Then $P_M x = \sum_{i=1}^n \lambda_i e_i$, $\{\lambda_i\}_{i=1}^n \subseteq \mathbb{F}$, and

$$\langle x, e_i \rangle \overset{e_i \in M}{=\!=} \langle x, P_M e_i \rangle \overset{P_M = P_M^*}{=\!=\!=} \langle P_M x, e_i \rangle = \langle \textstyle\sum_{j=1}^n \lambda_j e_j, e_i \rangle$$

$$= \textstyle\sum_{j=1}^n \lambda_j \langle e_j, e_i \rangle \overset{\langle e_j, e_i \rangle = \delta_{i,j}}{=\!=\!=} \lambda_i.$$

Hence $P_M x = \sum_{i=1}^n \langle x, e_i \rangle e_i$ and

$$\|P_M x\|^2 = \langle \sum_{i=1}^n \lambda_i e_i, \sum_{j=1}^n \lambda_j e_j \rangle = \sum_{i,j=1}^n \lambda_i \overline{\lambda}_j \langle e_i, e_j \rangle \overset{\langle e_j, e_i \rangle = \delta_{i,j}}{=\!=\!=} \sum_{i=1}^n |\lambda_i|^2$$

$$= \sum_{i=1}^n |\langle x, e_i \rangle|^2. \qquad \blacksquare$$

Cor. (Bessel inequality)

For any orthonormal system $\{e_i\}_{i\in I} \subseteq H$ we have

$$\sum_{i\in I} |\langle x, e_i\rangle|^2 \leqslant \|x\|^2, \qquad x \in H.$$

Proof: Let $A \subseteq I$ be finite and let $M = \operatorname{lin}\{e_i : i \in A\}$ be the linear space spanned by $\{e_i\}_{i\in A}$. Then

$$\sum_{i\in A} |\langle x, e_i\rangle|^2 \overset{Prop}{=\!=} \|P_M x\|^2 \overset{\|P_M\|\leqslant 1}{\leqslant} \|x\|^2.$$

Hence $\sum_{i\in I} |\langle x, e_i\rangle|^2 = \sup_{\substack{A\subseteq I \\ \text{finite}}} \sum_{i\in A} |\langle x, e_i\rangle|^2 \leqslant \|x\|^2$. ∎

Def. For a family of vectors $\{x_i\}_{i\in I}$ in a normed space X, the series $\sum_{i\in I} x_i$ is **(unconditionally) convergent** to a vector $x \in X$, if for every $\varepsilon > 0$ there is finite $K \subseteq I$ such that for any finite $J \subseteq I$ containing K we have $\|\sum_{i\in J} x_i - x\| < \varepsilon$. We then write $x = \sum_{i\in I} x_i$.

Def. An orthonormal basis of a Hilbert space H is an orthonormal system $\{e_i\}_{i\in I}$, which is maximal, i.e. there is no $e \in H$ such that the system $\{e_i\}_{i\in I} \cup \{e\}$ is orthonormal.

Prop. Every orthonormal system can be extended to an orthonormal basis. In particular, every Hilbert space has an orthonormal basis.

Proof: This follows from the Kuratowski-Zorn Lemma.

Indeed, let $\{e_i\}_{i\in I} \subseteq H$ be a fixed orthonormal system. Let $\mathcal{P}$ be the family of all orthonormal systems $\{e'_i\}_{i\in I'} \subseteq H$ that extend $\{e_i\}_{i\in I}$, that is $\{e_i : i \in I\} \subseteq \{e'_i : i \in I'\}$. This is a partially ordered set with respect to the inclusion relation. Moreover, every family of orthonormal systems $\mathcal{C} \subseteq \mathcal{P}$, which is linearly ordered (i.e. if $u, u' \in \mathcal{C}$, then either $u \subseteq u'$ or $u' \subseteq u$) has an upper bound $\bigcup_{u\in\mathcal{C}} u$. Therefore, by the Kuratowski–Zorn Lemma, there exists a maximal element in $\mathcal{P}$. ∎

Thm. (Characterizations of the orthonormal basis)

Let $\{e_i\}_{i\in I} \subseteq H$ be an orthonormal system in space Hilbert H. The following conditions are equivalent:

1. $\{e_i\}_{i\in I}$ is an **orthonormal basis**, i.e. $\{e_i\}_{i\in I}$ is a maximal orthonormal system.
2. the family $\{e_i\}_{i\in I}$ is **linearly dense** in H, i.e. $\overline{\operatorname{lin}\{e_i : i \in I\}} = H$.
3. For every $x \in H$, $\|x\|^2 = \sum_{i\in I} |\langle x, e_i\rangle|^2$, i.e. Bessel's inequality becomes the equality (called **Parseval's equality**).
4. Every $x \in H$ is of the form $x = \sum_{i\in I} \lambda_i e_i$, where $\lambda_i \in \mathbb{F}$, $i \in I$.

The numbers $\{\lambda_i\}_{i\in I} \subseteq \mathbb{F}$ in (4) are uniquely deterenined by x, namely $\lambda_i = \langle x, e_i\rangle$ for $i \in I$. We call these numbers **Fourier coefficients** of x in basis $\{e_i\}_{i\in I}$.

Proof: Note that if $x = \sum_{i\in I} \lambda_i e_i$ for some $\lambda_i \in \mathbb{F}$, then

$$\langle x, e_i\rangle = \langle \sum_{j\in I} \lambda_j e_j, e_i\rangle = \sum_{j\in I} \lambda_j \langle e_j, e_i\rangle = \sum_{j\in I} \lambda_j \delta_{i,j} = \lambda_i, \qquad i \in I.$$

This proves the last part of the assertion.

(1)⇒(2). Let $M := \overline{\mathrm{lin}\{e_i : i \in I\}}$ be a closed subspace of H generated by $\{e_i\}_{i\in I}$. If $M \neq H$, then $M^\perp \neq \{0\}$, so there is $e \in M^\perp$ with $\|e\| = 1$. Then $\{e_i\}_{i\in I} \cup \{e\}$ is an orthonormal system, which contradicts maximality of $\{e_i\}_{i\in I}$. Hence $M = H$.

(2)⇒(3). Let $x \in H$. Fix $\varepsilon > 0$. By assumption there is finite $A \subseteq I$ and a vector $v \in M := \mathrm{lin}\{e_i : i \in A\}$ such that $\|x - v\| < \varepsilon$. Note that $\dim(M) = |A| < \infty$. Thus

$$\begin{aligned}\|x\|^2 = \|P_M x + (x - P_M x)\|^2 &\overset{\text{Pitagoras}}{=} \|P_M x\|^2 + \|x - P_M x\|^2 \\ &\leqslant \|P_M x\|^2 + \|x - v\|^2 \quad \left\{ \begin{smallmatrix} \|x - P_M x\| = \\ \inf_{y\in M} \|x-y\| \end{smallmatrix} \right\} \\ &< \|P_M x\|^2 + \varepsilon^2 \overset{\text{wzór na } P_m}{=} \sum_{i\in A} |\langle x, e_i\rangle|^2 + \varepsilon^2 \\ &\leqslant \sum_{i\in I} |\langle x, e_i\rangle|^2 + \varepsilon^2.\end{aligned}$$

Passing with ε to zero we get that $\|x\|^2 \leqslant \sum_{i\in I} |\langle x, e_i\rangle|^2$. This is the opposite of Bessel's inequality. Hence the equality.

(3)⇒(4). Let $x \in H$ and $\varepsilon > 0$. The series $\|x\|^2 = \sum_{i\in I} |\langle x, e_i\rangle|^2$ converges (by Parseval's identity). Hence there is a finite $K \subseteq I$ such that for every finite $J \subseteq I$ disjoint with K we have $\sum_{i\in J} |\langle x, e_i\rangle|^2 < \varepsilon$. Hence

$$\sum_{i\in I\setminus K} |\langle x, e_i\rangle|^2 \leqslant \varepsilon.$$

Take now any finite $J \subseteq I$ containing K. Note that

$$\langle x - \sum_{j\in J}\langle x, e_j\rangle e_j, e_i\rangle = \langle x, e_i\rangle - \sum_{j\in J}\langle x, e_i\rangle \delta_{i,j} = \langle x, e_i\rangle \cdot 1_{I\setminus J}(i).$$

Therefore, applying Parseval's formula to the vector $x - \sum_{j\in J}\langle x, e_j\rangle e_j$

$$\|x - \textstyle\sum_{j\in J}\langle x, e_j\rangle e_j\|^2 = \sum_{i\in I\setminus J} |\langle x, e_i\rangle|^2 \leqslant \sum_{i\in I\setminus K} |\langle x, e_i\rangle|^2 \leqslant \varepsilon.$$

This shows that $x = \sum_{i\in I}\langle x, e_i\rangle e_i$.

(4)⇒(1). Let's assume ad absurdum that there is $e \in H$ such that the system $\{e_i\}_{i\in I} \cup \{e\}$ is orthonormal. By assumption $e = \sum_{i\in I} \lambda_i e_i$ for $\lambda_i \in \mathbb{F}$, $i \in I$. But $\lambda_i = \langle e, e_i\rangle = 0$ for every $i \in I$. Hence $e = 0$, which leads to the contradiction with the condition $\|e\| = 1$. ■

Cor. If $\{e_i\}_{i\in I}$ is an orthonormal basis of the Hilbert space H, then every $x \in H$ is uniquely determined by its Fourier coefficients $\{\langle x, e_i\rangle\}_{i\in I}$ via the **Fourier series**

Joseph Fourier

$$x = \sum_{i\in I} \langle x, e_i\rangle e_i.$$

Moreover, $\|x\| = \sqrt{\sum_{i\in I} |\langle x, e_i\rangle|^2}$ (Parseval's identity).

Ex. (Standard basis in $\ell^2(I)$)

Let I be an arbitrary set. Consider the Hilbert space

$$\ell^2(I) := \{x : I \to \mathbb{F} : \sum_{i\in I} |x(i)|^2 < \infty\}.$$

The inner product on $\ell^2(I)$ is defined by the formula $\langle x, y\rangle = \sum_{i\in I} x(i)\overline{y(i)}$, and the standard orthonormal basis is given by $\{e_i\}_{i\in I}$, where $e_i(j) = \delta_{i,j}$, for $j \in I$. If $I = \mathbb{N}$, then $\ell^2(\mathbb{N}) = \ell^2$ and

$$e_n = (\underbrace{0, \dots, 0}_{n-1}, 1, 0, \dots), \qquad n \in \mathbb{N}.$$

Thm. If $\{e_i\}_{i \in I}$ is an orthonormal basis of the Hilbert space H, then the formula $(Ux)(i) := \langle x, e_i \rangle$ dla $x \in H$, $i \in I$, defines an isometric isomorphism $U : H \to \ell^2(I)$ (**unitary operator**). Hence $H \cong \ell^2(I)$.

Proof: Parseval's equality implies that $U : H \to \ell^2(I)$ is a well defined isometry. Linearity is straightforward. Surjectivity 

Def. Dimension of a Hilbert space H is the cardinality of an orthonormal basis. We denote it by $\dim(H)$.

Rem. The dimension of a Hilbert space is well defined, i.e. any two bases of the same space have the same cardinality.


This follows from the Cantor–Bernstein Theorem ( read).

Cor. Two Hilbert spaces H and K are isometrically isomorphic if and only if $\dim(H) = \dim(K)$.